

Gauge Freedom in complex holomorphic systems

Carlos A. Margalli and J. David Vergara

Instituto de Ciencias Nucleares,
Universidad Nacional Autónoma de México,
A. Postal 70-543 , México D.F., México

E-mail: `carlos.margalli@nucleares.unam.mx`

E-mail: `vergara@nucleares.unam.mx`

january, 23 2017

Abstract. The aim of this paper is to introduce and analyze a new gauge symmetry that appears in complex holomorphic systems. This symmetry allow us to project the system, using different gauge conditions, to several real systems which are connect by gauge transformations in the complex space. We prove that the space of solutions of one system is related to the other by the gauge transformation. The gauge transformations are in some cases canonical transformations. However, in other cases are more general transformations that change the symplectic structure, but there is still a map between the systems. In this way our construction extend the group of canonical transformations in classical mechanics. Also, we show how to extend the analysis to the quantum case using path integrals by means of the Batalin-Fradkin-Vilkovisky theorem and within the canonical formalism, where we show explicitly that solutions of the Schrödinger equation are gauge related.

PACS numbers: 02.30.Fn, 03.65.-w, 11.15.-q, 11.10.Nx

Keywords: Gauge Theories, Non-Commutative Theory, Mapping Between Solutions
 Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

Symmetries in physical systems play an important role in solving and understanding a system. A very useful tool to describe local symmetries is the Dirac-Bergmann theory of constrained Hamiltonian systems [1, 2]. In this theory the local generators of gauge symmetries are first class constraints, the existence of these constraints imply that the dynamics of some degrees of freedom are not determined by the equations of motion. However, it is possible to determine the complete dynamics of the system by removing the remaining degrees of freedom by introducing a canonical gauge condition that satisfy certain requirements in combination with the first class constraints [3]. In addition, it is

well known that the theories obtained by using different gauge conditions are equivalent and are related by a gauge transformation and the physics is gauge invariant.

On the other hand, in Quantum Mechanics and Quantum Field Theory is quite natural to extend the symmetry transformations from the real plane to the complex plane, and in fact, there has been attempts to raise theories with physical sense based on the structure of complex functions [4, 5, 6].

In this work, we consider the extension to the complex plane of any Lagrangian, with the condition that the extension is given only in terms of holomorphic variables. This condition implies, that there exists a hidden gauge symmetry in the theory that allow us to project to the real physical space [7, 8], in quite different ways, according to the gauge choice. Furthermore, these real theories are related under a gauge transformation in the complex space.

Moreover, the extension on the complex plane of operators resulting from the Quantum Mechanics is quite important considering the Hermiticity hypothesis. Using the creation and annihilation operators, we can implement linear canonical transformations [9] that map between unitary equivalent theories. In our case, we extend this concept since the gauge transformation that relates two systems it is not necessarily canonical or unitary. For example, the complex canonical transformation introduced by Ashtekar is in some sense a very particular case of our work. Since, in [10] was shown that the reality conditions can be interpreted as second class constraints, then this set of constraints can be interpreted as a pair of a first class constraint that generates the gauge transformations and the corresponding gauge fixing condition.

In this paper, we analyze this gauge symmetry, we show that the temporal evolution of the primary constraints is a manifestation of the Cauchy-Riemann equations [11], then if we impose that our Lagrangian is an holomorphic function, these equations are satisfied identically and the primary constraints result a set of generators of gauge transformations. As a second step we impose several sets of gauge conditions that imply different projections to the real space. The resulting theories are not related by a canonical transformation, however are related by a gauge transformation in the complex space and we show how from the solution of one theory we can obtain the solution of the other gauge related theory by using the complex gauge transformation. This gauge transformation is implemented in the classical and the quantum cases.

The organization of this article is as follows, in Section 2, we consider a complex two-dimensional model, and we show that there is a trivial gauge symmetry that leaves the system invariant. Using Dirac's method [1], in the space separated in real and imaginary parts we show that associated with the symmetry there are two first class constraints. In consequence, we can fix the gauge. Using two different sets of gauge conditions we map the initial complex space to two different real systems. A case corresponds to two-dimensional harmonic oscillator and the other to a Hamiltonian of a free particle in a non-constant metric and a nontrivial symplectic structure. Solving the dynamics of both systems we show that they are related by a complex gauge transformation in the original complex phase space.

In Section 3 we introduce a gauge condition that lead us to a non-commutative theory but this theory is gauge related to the two-dimensional harmonic oscillator, with the usual symplectic structure. We explicitly build the gauge transformations and we show how the solutions of one system are connected with the solutions of the other theory. Section 4 is devoted to analyze the relation between two gauge related quantum theories, to that end we use the BFM formalism [12] and we prove by using two different gauges how we can go from one theory to the other. In Section 6 we implement our procedure in the quantum canonical case, using a parametrized system. We fix partially the gauge and proof that the solutions of two gauge related Schrödinger equations are gauge related. Finally Section 7 contains a brief discussion of our results.

2. Gauge Conditions For The Case Of Two Dimensions

The aim of this section is to describe the general basis of our procedure. Specifically, we consider an harmonic oscillator with complex variables in two dimensions that has a gauge freedom. Fixing the gauge it is possible to obtain a real system. The real system depends on the gauge conditions selected and for our study it is important to consider, the usual gauge conditions that reduce to a real harmonic oscillator in two dimensions, the gauge conditions where we have a linear dependence on momenta, and the gauge conditions where we have quadratic dependence on momenta generating Dirac's brackets for position variables that are no trivial.

The complex model written in terms of a Lagrangian is

$$L(z_1, z_2) = \frac{1}{2}\dot{z}_1^2 + \frac{1}{2}\dot{z}_2^2 - \frac{\omega_1^2}{2}z_1^2 - \frac{\omega_2^2}{2}z_2^2 \quad (1)$$

with $z_1 = x + iy$ and $z_2 = u + iv$. Here it is necessary to mention that the above Lagrangian is invariant, if we apply the transformation

$$\begin{aligned} x' &= x + \lambda(t), & y' &= y + i\lambda(t). \\ u' &= u + \bar{\lambda}(t), & v' &= v + i\bar{\lambda}(t) \end{aligned} \quad (2)$$

where $\bar{\lambda}(t), \lambda(t)$ are arbitrary functions of the time, then we have a local symmetry. Notice that we are considering an extension of the usual allowed transformations, since the real functions y and v get an imaginary part. However, we obtain

$$z'_1 = z_1, \quad z'_2 = z_2, \quad L' = L, \quad (3)$$

and in this way we get a legitimate symmetry. Now, we proceed to the canonical analysis of this symmetry. From the above Lagrangian, we obtain the momenta

$$\begin{aligned} p_x &= \dot{x} + i\dot{y}, & p_y &= -\dot{y} + i\dot{x}, \\ p_u &= \dot{u} + i\dot{v}, & p_v &= -\dot{v} + i\dot{u}, \end{aligned} \quad (4)$$

from the definition of these momenta we observe that there are two primary constraints

$$\begin{aligned} \Phi_0 &= p_x + ip_y \approx 0, \\ \Phi_1 &= p_u + ip_v \approx 0. \end{aligned} \quad (5)$$

On another hand, if we take the Lagrangian (1), we compute the canonical Hamiltonian, using the ordinary Lagrange transform, that results

$$H_{2C} = p_x \dot{x} + p_y \dot{y} + p_u \dot{u} + p_v \dot{v} - L \quad (6)$$

$$= \frac{1}{2} p_x^2 + \frac{\omega_1^2}{2} (x + iy)^2 + \frac{1}{2} p_u^2 + \frac{\omega_2^2}{2} (u + iv)^2. \quad (7)$$

The symplectic structure of our system is given by the fundamental Poisson brackets

$$\{x, p_x\} = 1, \quad \{y, p_y\} = 1, \quad \{u, p_u\} = 1, \quad \{v, p_v\} = 1. \quad (8)$$

Notice that the Hamiltonian (7) is written only in terms of the momenta p_x and p_u , this is essentially due to the constraints (5), since the momenta p_y and p_v can be eliminated. The total Hamiltonian for the system is

$$H_{2T} = H_{2C} + \mu^a \Phi_a, \quad (9)$$

with $a = 1, 2$ and μ^a are arbitrary Lagrange multipliers. Using this Hamiltonian the temporal evolution for the primary constraints are given by

$$\dot{\Phi}_0 = \{\Phi_0, H_{2T}\} = 0, \quad \dot{\Phi}_1 = \{\Phi_1, H_{2T}\} = 0, \quad (10)$$

saying what the primary constraints are first class constraints and then generators of gauge transformations. The gauge transformations of our variables are given by

$$\delta x = \{x, \epsilon^a \Phi_a\} = \epsilon^1, \quad \delta y = \{y, \epsilon^a \Phi_a\} = i\epsilon^1, \quad (11)$$

$$\delta u = \{u, \epsilon^a \Phi_a\} = \epsilon^2, \quad \delta v = \{v, \epsilon^a \Phi_a\} = i\epsilon^2, \quad (12)$$

Here we notice that these transformations are the infinitesimal form of the finite Lagrange transformations (2). Due the gauge freedom in the system, we need to fix gauge conditions, our first choice is

$$\gamma_0^1(y) = y \approx 0, \quad \gamma_1^1(v) = v \approx 0, \quad (13)$$

and the brackets between the constraints are

$$\{\Phi_0, \gamma_0^1\} = -i, \quad \{\Phi_1, \gamma_1^1\} = -i. \quad (14)$$

Now it is possible to make a complete set of second class constraints $\chi_a = (\phi_0, \gamma_0^1, \phi_1, \gamma_1^1)$ with the matrix of the second class constraints given by

$$\mathcal{A}_{ab} = \{\chi_a, \chi_b\} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \quad (15)$$

that has a determinanrt different of zero

$$\det A_{ab} = 1 \quad (16)$$

On another hand, we have a reduced Hamiltonian as a consequence of applying the set of constraints

$$H_{HO2} = \frac{1}{2} p_{\bar{x}}^2 + \frac{\omega_1^2}{2} \bar{x}^2 + \frac{1}{2} p_{\bar{u}}^2 + \frac{\omega_2^2}{2} \bar{u}^2. \quad (17)$$

Here, it is necessary to establish a change of variables in order to avoid confusions. For these ones and only for these conditions, we define:

$$x = \bar{x}, \quad y = \bar{y}, \quad (18)$$

and now, we can compute the Dirac brackets from Hamiltonian theory that are reduced to

$$\{\bar{x}, p_{\bar{x}}\}^* = 1, \quad \{\bar{u}, p_{\bar{u}}\}^* = 1, \quad (19)$$

with other combinations of brackets, being zero and resulting an harmonic oscillator in two dimensions. Using these brackets it is straightforward to obtain the equations of motion in usual way

$$\begin{aligned} \dot{\bar{x}} &= \{\bar{x}, H_{HO2}\}^* = p_{\bar{x}}, & \dot{\bar{u}} &= \{\bar{u}, H_{HO2}\}^* = p_{\bar{u}}, \\ \dot{p}_{\bar{x}} &= \{p_{\bar{x}}, H_{HO2}\}^* = -\omega_1^2 \bar{x}, & \dot{p}_{\bar{u}} &= \{p_{\bar{u}}, H_{HO2}\}^* = -\omega_2^2 \bar{u}, \end{aligned} \quad (20)$$

with solution given by

$$\bar{x} = A_0 \sin(\omega_1 t + \delta_0), \quad \bar{u} = A_1 \sin(\omega_2 t + \delta_1), \quad (21)$$

where it must be mentioned that the free parameters are fixed with initial conditions, being $A_0, \delta_0, A_1, \delta_1$.

In another hand, these gauge conditions are not special and they could be selected in other way to obtain a different system, but both theories are gauge related on some region of the phase space. A particular election is to consider the gauge conditions that include linear terms of the momenta

$$\begin{aligned} \gamma_{0*} &= y - ix + ig_2(x)p_x \approx 0, \\ \gamma_{1*} &= v - iu + ih_2(u)p_u \approx 0, \end{aligned} \quad (22)$$

Then our set of second class constraints results $\chi_A = (\Phi_0, \gamma_{0*}, \Phi_1, \gamma_{1*})$. By defining

$$B = -ip_x \partial_x g_2(x), \quad C = -ip_u \partial_u h_2(u), \quad (23)$$

in order to calculate easily the constraint matrix of second class $\mathcal{B}_{AB} = \{\chi_A, \chi_B\}$. Associated to this set we obtain

$$\mathcal{B}_{ab} = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & -C & 0 \end{pmatrix}. \quad (24)$$

The determinant for (24) must be different of zero in order to get a good set of constraints. At this point, it is necessary to consider a concrete example of (22) and to choose specifically the functions

$$g_2(x) = x, \quad h_2(u) = u, \quad (25)$$

with an associated determinant for (24) and (25)

$$\det B_{ab} = B^2 C^2 = (p_x \partial_x g_2)^2 (p_u \partial_u h_2)^2 = p_x^2 p_u^2. \quad (26)$$

On another hand, we obtain the reduced Hamiltonian using these gauge conditions

$$\begin{aligned} H_{cons} &= \left[\frac{1}{2} + \frac{\omega_1^2}{2} g_2^2(x) \right] p_x^2 + \left[\frac{1}{2} + \frac{\omega_2^2}{2} h_2^2(u) \right] p_u^2 \\ &= \left[\frac{1}{2} + \frac{\omega_1^2}{2} x^2 \right] p_x^2 + \left[\frac{1}{2} + \frac{\omega_2^2}{2} u^2 \right] p_u^2 = g^{\mu\nu} \frac{p_\mu p_\nu}{2}, \end{aligned} \quad (27)$$

where, we see that it is equivalent to a free particle in the metric

$$g_{\mu\nu} = \begin{pmatrix} \frac{2}{1+\omega_1^2 x^2} & 0 \\ 0 & \frac{2}{1+\omega_2^2 u^2} \end{pmatrix}, \quad (28)$$

and from this expression we obtain that the curvature $R = 0$. However, the dynamics of the system is not only given by the Hamiltonian, we need to take into account that the symplectic structure is given in the reduced space by the Dirac brackets

$$\{x, p_x\}^* = \frac{1}{p_x \partial_x g_2(x)} = \frac{1}{p_x}, \quad \{u, p_u\}^* = \frac{1}{p_u \partial_u h_2(u)} = \frac{1}{p_u}, \quad (29)$$

and zero for other combinations. Through the above brackets (29) we determine the temporal evolution for the phase space variables

$$\begin{aligned} \dot{x} &= \{x, H_{cons}\}^* = \frac{(1 + \omega_1^2 g_2^2(x))}{\partial_x g_2(x)} = (1 + \omega_1^2 x^2), \\ \dot{p}_x &= \{p_x, H_{cons}\}^* = -\omega_1^2 g_2(x) p_x = -\omega_1^2 x p_x, \\ \dot{u} &= \{u, H_{cons}\}^* = \frac{(1 + \omega_2^2 h_2^2(u))}{\partial_u h_2(u)} = (1 + \omega_2^2 u^2), \\ \dot{p}_u &= \{p_u, H_{cons}\}^* = -\omega_2^2 h_2(u) p_u = -\omega_2^2 u p_u, \end{aligned} \quad (30)$$

and we obtain an expression associated to the momenta in this reduced theory

$$\begin{aligned} p_x &= p_{x0} \exp \left(-\omega_1^2 \int_{t_0}^t d\tau g_2(x) \right) = p_{x0} \exp \left(-\omega_1^2 \int_{t_0}^t d\tau x \right), \\ p_u &= p_{u0} \exp \left(-\omega_2^2 \int_{t_0}^t d\tau h_2(u) \right) = p_{u0} \exp \left(-\omega_2^2 \int_{t_0}^t d\tau u \right). \end{aligned} \quad (31)$$

Now it is possible to establish a reduced theory, as long as we select the functions $g_2(x), h_2(u)$. Using (25) it is possible to solve the equations of motion and obtain values for p_x and p_u once we have obtained solutions for x and u .

The solutions obtained from equations (30) are

$$\begin{aligned} x &= \frac{1}{\omega_1} \tan [\omega_1(t - t_0) + \tan^{-1}(\omega_1 x_0)], \\ u &= \frac{1}{\omega_2} \tan [\omega_2(t - t_0) + \tan^{-1}(\omega_2 u_0)], \end{aligned} \quad (32)$$

and for the momenta we get

$$\begin{aligned} p_x &= p_{x_0}(1 + \omega_1^2 x_0^2)^{\frac{1}{2}} \cos[\omega_1(t - t_0) + \tan^{-1}(\omega_1 x_0)], \\ p_u &= p_{u_0}(1 + \omega_2^2 u_0^2)^{\frac{1}{2}} \cos[\omega_2(t - t_0) + \tan^{-1}(\omega_2 u_0)]. \end{aligned} \quad (33)$$

On another hand, it is important to point out that there are points where the mapping does not apply. These points correspond to determinant (26) equal to zero, that imply that squared momenta vanish (33). Furthermore, the momenta are periodic functions.

2.1. Solutions Using Gauge Transformation

In this subsection, we will show you how to solve the above system through a gauge transformation. Let us consider the gauge conditions (13) and (22), then the difference between these transformations parametrize the gauge transformation between the systems (17) and (27)

$$\begin{aligned} \delta\gamma_0 &= \gamma_{0*} - \gamma_0^1 = -ix + ig_2(x)p_x \approx 0, \\ \delta\gamma_1 &= \gamma_{1*} - \gamma_1^1 = -iu + ih_2(u)p_u \approx 0. \end{aligned} \quad (34)$$

By applying these gauge transformations to our variables we get

$$\begin{aligned} \delta x &= x - \bar{x} = i\delta\gamma_0\{x, \Phi_0\}, & \delta u &= u - \bar{u} = i\delta\gamma_1\{u, \Phi_1\}, \\ \delta p_x &= 0, & \delta p_u &= 0, \end{aligned} \quad (35)$$

where, we can infer using (25),

$$\bar{x} = g_2(x)p_x = xp_x, \quad \bar{u} = h_2(u)p_u = up_u, \quad p_{\bar{x}} = p_x, \quad p_{\bar{u}} = p_u \quad (36)$$

Where we are considering that the variables with bar correspond to the case (13) and the variables without bar to the gauge (22). Now, starting from the equations (20)

$$\begin{aligned} \dot{\bar{x}} &= \frac{d}{dt}(xp_x) = p_{\bar{x}} = p_x, & \dot{\bar{u}} &= \frac{d}{dt}(up_u) = p_{\bar{u}} = p_u, \\ \dot{p}_{\bar{x}} &= \dot{p}_x = -\omega_1^2 \bar{x}, & \dot{p}_{\bar{u}} &= \dot{p}_u = -\omega_2^2 \bar{u}, \end{aligned} \quad (37)$$

by rewriting these equations in terms of variables without bar we get

$$\dot{p}_x = -\omega_1^2 xp_x, \quad \dot{p}_u = -\omega_2^2 up_u, \quad (38)$$

$$\dot{x} = \frac{(1 + \omega_1^2 g_2^2(x))}{\partial_x g_2(x)} = (1 + \omega_1^2 x^2), \quad \dot{u} = \frac{(1 + \omega_2^2 h_2^2(u))}{\partial_u h_2(u)} = (1 + \omega_2^2 u^2), \quad (39)$$

where it is important to mention that using the equations (38) we obtain equations (39) through the gauge transformations (36). Furthermore, the equations coincide with the equations (30).

Additionally, the solutions of both systems have a relationship associated to the gauge transformation (36). By applying to the solution of harmonic oscillator (21), the gauge transformation (36) we obtain

$$\begin{aligned} x &= \frac{\bar{x}}{p_x} = \frac{\sin(\omega_1(t - t_0) + \delta_0)}{\omega_1 \cos(\omega_1(t - t_0) + \delta_0)} = \frac{\tan(\omega_1(t - t_0) + \delta_0)}{\omega_1}, \\ u &= \frac{\bar{u}}{p_u} = \frac{\sin(\omega_2(t - t_0) + \delta_1)}{\omega_2 \cos(\omega_2(t - t_0) + \delta_1)} = \frac{\tan(\omega_2(t - t_0) + \delta_1)}{\omega_2}, \end{aligned} \quad (40)$$

that is equivalent to (32) except a phase difference, that is fixed by means of selecting a initial condition that determines the interval where is valid the solution.

We must notice that, this gauge transformation has determinant equal to zero for a time given by $\frac{n\pi}{2\omega_{1,2}}$ when n is odd, see equation (26). This is linked to zones where the mapping is valid.

On another hand, because first class constraints only depend from momenta. Momenta from harmonic oscillator coincide with momenta from the theory resulting of gauge conditions that are linear on momenta (22). However, the initial conditions are different and it is possible to use equations (38) in order to find momenta (33). With respect to the solutions of both systems and the relationship between initial conditions, we find

$$\begin{aligned}\delta_0 &= \tan^{-1}(x_0), & \delta_1 &= \tan^{-1}(u_0), \\ A_0 &= \frac{1}{\omega_1} p_{x_0} (1 + \omega_1^2 x_0^2)^{\frac{1}{2}}, & A_1 &= \frac{1}{\omega_2} p_{u_0} (1 + \omega_2^2 u_0^2)^{\frac{1}{2}},\end{aligned}\quad (41)$$

that are invertible in an temporal interval given by the tangent function. In the next section we generalize this idea to other gauge conditions.

3. Quadratic Gauge Condition In Momenta

Let's consider another kind of gauge conditions depending on quadratic form of the momenta

$$\begin{aligned}\gamma_{0+} &= y - ix + ig_2(x)p_x p_u, \\ \gamma_{1+} &= v - iu + ih_2(u)p_u.\end{aligned}\quad (42)$$

As an important element of this work and in order to make easy the notation, we define the next quantities

$$\begin{aligned}A_1 &= -ip_x p_u \partial_x g_2(x), & B_1 &= -g_2(x)p_x + p_x p_u g_2(x) \partial_u h_2(u), \\ E_1 &= -ip_u \partial_u h_2(u),\end{aligned}\quad (43)$$

in such a way that we establish a set of second class constraints $\chi_a = (\Phi_0, \gamma_{0+}, \Phi_1, \gamma_{1+})$ that allows us to find a constraint matrix by means of the Poisson brackets $\mathcal{G}_{ab} = \{\chi_a, \chi_b\}$ resulting

$$\mathcal{G}_{ab} = \begin{pmatrix} 0 & A_1 & 0 & 0 \\ -A_1 & 0 & 0 & B_1 \\ 0 & 0 & 0 & E_1 \\ 0 & -B_1 & -E_1 & 0 \end{pmatrix}.\quad (44)$$

Furthermore the associated determinant is

$$\det \mathcal{G}_{ab} = A_1^2 E_1^2 = p_x^2 p_u^4 [\partial_x g_2(x)]^2 [\partial_u h_2(u)]^2, \quad (45)$$

and it must be different from zero.

Here, we choose a concrete case and assign a value for g_2 and h_2

$$g_2(x) = \frac{\theta}{2}x, \quad h_2(u) = u, \quad (46)$$

that allows to find a value for the determinant

$$\det(G_{ab}) = \frac{\theta^2}{4} p_x^2 p_u^4, \quad (47)$$

which depends from powers of momenta. Furthermore, the reduced Hamiltonian is given by

$$H_{1Cons} = [\frac{1}{2} + \frac{\omega_1^2}{2} g_2^2(x) p_u^2] p_x^2 + [\frac{1}{2} + \frac{\omega_2^2}{2} h_2^2(u)] p_u^2 \quad (48)$$

and using (46) the Hamiltonian is

$$H_{1Cons} = [\frac{1}{2} + \frac{\omega_1^2}{2} (\frac{\theta}{2} x)^2 p_u^2] p_x^2 + [\frac{1}{2} + \frac{\omega_2^2}{2} u^2] p_u^2. \quad (49)$$

Now, the symplectic structure of the system is given by the Dirac brackets

$$\begin{aligned} \{x, p_x\}^* &= \frac{1}{p_x p_u \partial_x g_2(x)} = \frac{2}{\theta p_x p_u}, \\ \{x, u\}^* &= \frac{g_2(x)}{p_u^2 \partial_x g_2(x) \partial_u h_2(u)} = \frac{x}{p_u^2}, \\ \{u, p_u\}^* &= \frac{1}{p_u \partial_u h_2(u)} = \frac{1}{p_u}, \end{aligned} \quad (50)$$

while the other brackets vanish. We observe from (50) that in this case the symplectic structure gives rise to a non-commutative theory. In addition, we compute the temporal evolution of x and u resulting

$$\begin{aligned} \dot{x} &= \{x, H_{1Cons}\}^* = \\ &= \frac{(1 + \omega_1^2 g_2^2(x) p_u^2)}{\partial_x g_2(x) p_u} + \omega_2^2 x h_2(u) \partial_u h_2(u) = \frac{2(1 + \omega_1^2 (\theta^2/4) x^2 p_u^2)}{\theta p_u} + \omega_2^2 x u, \\ \dot{u} &= \{u, H_{1Cons}\}^* = (1 + \omega_2^2 h_2^2(u)) = (1 + \omega_2^2 u^2), \end{aligned} \quad (51)$$

and we observe that using the first equation it is possible to determine the momentum p_u , but from the second equation we obtain a relationship that does not determine p_x . However, we can compute $u(t)$ through $h_2(u) = u$.

From the temporal evolution for p_x and p_u , we obtain all the Hamilton equations of motion

$$\dot{p}_x = -\omega_1^2 g_2(x) p_x p_u = -\frac{\omega_1^2}{2} \theta x p_x p_u, \quad \dot{p}_u = -\omega_2^2 h_2(u) p_u = -\omega_2^2 u p_u. \quad (52)$$

3.1. Gauge Transformation with Non Trivial Dirac's Brackets for Positions

Now, we are looking for the solution of the equations of motion (51) and (52) by using the gauge transformations and its relation with the harmonic oscillator problem. We know that the parameters of the gauge transformations between the gauges (13) and the gauges (42) are given to first order by the differences

$$\begin{aligned} \delta\gamma_{0+} &= \gamma_{0+} - \gamma_0 = -ix + ig_2(x) p_x p_u = -ix + i\frac{\theta x}{2} p_x p_u, \\ \delta\gamma_{1+} &= \gamma_{1+} - \gamma_1 = -iu + ih_2(u) p_u = -iu + iu p_u. \end{aligned} \quad (53)$$

In accordance with the above, we establish the gauge transformations that take us from the theory with Hamiltonian (17) and symplectic structure (19) to the theory with Hamiltonian (49) with symplectic structure (50), that are given by

$$\begin{aligned}\delta x &= \bar{x} - x = -i\delta\gamma_{0+}\{x, \Phi\} = -x + g_2(x)p_x p_u, \\ \delta u &= \bar{u} - u = -i\delta\gamma_1\{u, \Phi\} = -u + ih_2(u)p_u, \\ \delta p_x &= 0, \quad \delta p_u = 0,\end{aligned}\tag{54}$$

where we can deduce

$$\bar{x} = g_2(x)p_x p_u, \quad \bar{u} = h_2(u)p_u, \quad p_{\bar{x}} = p_x, \quad p_{\bar{u}} = p_u.\tag{55}$$

Then, we want to show that starting from the equations of motion for the harmonic oscillator (20) and using the gauge transformation (54) that imply (55), we can obtain the equations of motion (51) and (52), where we use that $g_2(x) = \frac{\theta x}{2}$ and $h_2(u) = u$, resulting

$$\dot{p}_x = -\omega_1^2 \bar{x} = -\omega_1^2 \frac{\theta x}{2} p_x p_u, \quad \dot{p}_u = -\omega_2^2 \bar{u} = -\omega_2^2 u p_u,\tag{56}$$

$$\dot{x} = \frac{2(1 + \omega_1^2(\theta^2/4)x^2 p_u^2)}{\theta p_u} + \omega_2^2 x u, \quad \dot{u} = (1 + \omega_2^2 u^2),\tag{57}$$

So, starting from equations of motion of two oscillators with usual symplectic structure, we arrive to a system of equations with non commutative variables. Similarly to the case in Section 2, both systems have a relationship through a gauge transformation, in this case given by (55). Now, if we want to solve the equations (56) and (57), we use the solution of the harmonic oscillator (21), and apply the gauge transformation to obtain

$$\begin{aligned}x &= \frac{2\bar{x}}{\theta p_x p_u} = \frac{2 \sin(\omega_1(t - t_0) + \delta_0)}{\omega_2 \omega_1 \theta \cos(\omega_1(t - t_0) + \delta_0) \cos(\omega_2(t - t_0) + \delta_1)} \\ &= \frac{2 \tan(\omega_1(t - t_0) + \delta_0)}{\omega_1 \omega_2 \theta \cos(\omega_2(t - t_0) + \delta_1)}, \\ u &= \frac{\bar{u}}{p_u} = \frac{\sin(\omega_2(t - t_0) + \delta_1)}{\omega_2 \cos(\omega_2(t - t_0) + \delta_1)} = \frac{\tan(\omega_2(t - t_0) + \delta_1)}{\omega_2},\end{aligned}\tag{58}$$

that except a constant and a difference of phase, it coincides with the solution of the system given by the equations (51) and (52).

On the other side, we can fix the solutions to the initial conditions of the non-commutative problem resulting

$$\begin{aligned}x &= \frac{2 \tan(\omega_1(t - t_0) + \tan^{-1}(\frac{\omega_1 \omega_2 \theta x_0}{2\sqrt{1 + \omega_2^2 u_0^2}}))}{\omega_1 \omega_2 \theta \cos(\omega_2(t - t_0) + \tan^{-1}(\omega_2 u_0))}, \\ u &= \frac{\tan(\omega_2(t - t_0) + \tan^{-1}(\omega_2 u_0))}{\omega_2},\end{aligned}\tag{59}$$

For the momenta the equivalence of the initial conditions is given by

$$\begin{aligned}
p_x &= p_{x_0} \left(1 + \frac{\omega_1^2 \omega_2^2 \theta^2 x_0^2}{4(1 + \omega_2^2 u_0^2)}\right)^{\frac{1}{2}} \cos(\omega_1(t - t_0) + \tan^{-1}(\frac{\omega_1 \omega_2 \theta x_0}{2\sqrt{1 + \omega_2^2 u_0^2}})), \\
p_u &= p_{u_0} (1 + \omega_2^2 u_0^2)^{\frac{1}{2}} \cos(\omega_2(t - t_0) + \tan^{-1}(\omega_2 u_0)).
\end{aligned} \tag{60}$$

Furthermore, the relationship between initial conditions for both systems, the harmonic oscillator in two dimensions (21) and the model with gauge condition non trivial, where Dirac's brackets for positions are not commutative, (59) and (60) is

$$\begin{aligned}
\delta_0 &= \tan^{-1}(\frac{\omega_1 \omega_2 \theta x_0}{2\sqrt{1 + \omega_2^2 u_0^2}}), & \delta_1 &= \tan^{-1}(\omega_2 u_0), \\
A_0 &= \frac{p_{x_0}}{\omega_1} \sqrt{1 + \frac{\omega_1^2 \omega_2^2 \theta^2 x_0^2}{4(1 + \omega_2^2 u_0^2)}}
\end{aligned} \tag{61}$$

In this way we have shown that using a gauge transformation induced in the complex space. We can solve different systems starting from the solution of the harmonic oscillator. Of course, the kind of systems that we can solve depend from the starting complex model and the gauge conditions imposed.

In next two sections, we want to implement these ideas to the quantum level, first using the path integral in the Batalin-Fradkin-Vilkovisky formalism BFV [13] and after that to the canonical level.

4. BFV Formalism In The Harmonic Oscillator In One Dimension

In this Section, we show at the quantum level, a relationship that exist between different gauge conditions and the way in that changes the corresponding quantum theory. To manage this problem we introduce the BRST formalism [12], in the formulation of BFV using as an example the complex harmonic oscillator in one dimension.

First of all, consider a formalism with one complex dimension and start from the Lagrangian

$$L = \frac{1}{2} \dot{z}^2 - \frac{\omega^2}{2} z^2 \tag{62}$$

where z is a complex variable that is separated into real and imaginary parts

$$z = x + iy. \tag{63}$$

The Lagrangian in terms of these variables is

$$L_{(x,y)} = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \dot{y}^2 + i\dot{x}\dot{y} - \frac{\omega^2}{2} x^2 + \frac{\omega^2}{2} y^2 - i\omega^2 xy \tag{64}$$

and we obtain the momenta

$$p_x = \dot{x} + i\dot{y}, \tag{65}$$

$$p_y = -\dot{y} + i\dot{x}. \tag{66}$$

We observe that these momenta are not independent, then generate the primary constraint

$$\Phi_0 = p_x + ip_y \approx 0. \quad (67)$$

The next step is to obtain the canonical Hamiltonian

$$H_0 = \dot{x}p_x + \dot{y}p_y - L_{(x,y)}, \quad (68)$$

and the total Hamiltonian will be

$$H_T = \frac{1}{2}p_x^2 + \frac{\omega^2}{2}x^2 - \frac{\omega^2}{2}y^2 + i\omega^2xy + \mu^0\Phi_0. \quad (69)$$

From the temporal evolution of (67) with the total Hamiltonian (69) we observe that the primary constraint is of first class and is the generator of a gauge symmetry.

In order to reduce the above theory, we must eliminate degrees of freedom by fixing a gauge condition that in general is a complex function, but its election is not unique.

Specifically, we choose two ways to fix the gauge $\gamma_{0,1}$ with

$$\begin{aligned} \gamma_0 &= y \approx 0, \\ \gamma_1 &= y - ix + iU^{\frac{1}{2}}(x), \end{aligned} \quad (70)$$

where the first election implies a real formulation of a harmonic oscillator. In contrast to the second gauge condition, that permits non trivial results

$$\{\Phi, \gamma_1\} = -\frac{i}{2U^{\frac{1}{2}}} \partial_x U(x).$$

Now we can use the BRST formalism. Here the phase space is extended including fermionic degrees of freedom. With these degrees of freedom, we establish the BRST symmetry that is a global remnant of a local gauge symmetry after the gauge fixing. The importance of a theory with these characteristics is based in that the propagator is a sum over all probabilities of process that involves only real particles on initial and final states and in consequence is unitary.

As the first step, we consider that the Lagrange multiplier μ^0 , in (69), is a variable of the configuration space with its respective momenta π_0 . For these variables, we associate the Poisson brackets

$$\{\mu^0, \pi_0\} = -\{\pi_0, \mu^0\} = 1, \quad (71)$$

which is part of an extended symplectic structure.

Additionally, it is necessary to include in the phase space a ghost \mathcal{C} and an anti-ghost $\bar{\mathcal{C}}$ with symplectic structure given by

$$\{\bar{\mathcal{P}}, \mathcal{C}\} = \{\mathcal{P}, \bar{\mathcal{C}}\} = -1, \quad (72)$$

and zero for another brackets. From these ghost and anti-ghost, we obtain the BRST charge, since can be seen that exists a new symmetry in this extended theory

$$\Omega = \mathcal{C}\Phi_0 - i\mathcal{P}\pi_0, \quad (73)$$

and the BRST charge Ω is the generator of the BRST symmetry. Furthermore, the gauge conditions are included through the fermionic conditions

$$\Psi_0 = i\bar{\mathcal{C}}\gamma_0 + \bar{\mathcal{P}}\mu^0, \quad \Psi_1 = i\bar{\mathcal{C}}\gamma_1 + \bar{\mathcal{P}}\mu^0, \quad (74)$$

resulting from the ghosts. Using these conditions and the BRST charge, we obtain a brackets that are used later to obtain the effective Hamiltonian

$$\begin{aligned} \{\Psi_0, \Omega\} &= -\mu^0\Phi_0 + \pi_0\gamma_0 - i\bar{\mathcal{P}}\mathcal{P} - \bar{\mathcal{C}}\mathcal{C}, \\ \{\Psi_1, \Omega\} &= -\mu^0\Phi_0 + \pi_0\gamma_1 - i\bar{\mathcal{P}}\mathcal{P} - \bar{\mathcal{C}}\mathcal{C}\left(\frac{\partial_x U}{2U^{\frac{1}{2}}}\right). \end{aligned} \quad (75)$$

From this structure is obtained a way to achieve the quantization by means of the path integral. Before it is necessary to show that exist a gauge freedom in the path integral and it is independent from the fermionic conditions (74), *i.e.* the Fradkin-Vilkovisky theorem [12].

The path integral using the BRST formalism and applied to the complex theory with a gauge condition Ψ_0 is

$$Z_\Psi = \int \mathcal{D}\mu \exp(iS_{eff}), \quad (76)$$

$$\begin{aligned} S_{eff} &= \int_{\tau_1}^{\tau_2} d\tau (\dot{x}p_x + \dot{y}p_y - \mu^0\dot{\pi}_0 \\ &\quad + \bar{\mathcal{C}}\dot{\mathcal{P}} + \bar{\mathcal{P}}\dot{\mathcal{C}} - H_{eff}), \end{aligned} \quad (77)$$

$$\begin{aligned} H_{eff} &= H_{BRST} - \{\Psi_0, \Omega\}, \\ H_{BRST} &= H_0, \end{aligned} \quad (78)$$

$$\begin{aligned} H_{eff} &= \frac{1}{2}p_x^2 + \frac{\omega^2}{2}x^2 - \frac{\omega^2}{2}y^2 + i\omega^2xy + \bar{\mathcal{C}}\mathcal{C} \\ &\quad + \mu^0\Phi_0 - \pi_0\gamma_0 + i\bar{\mathcal{P}}\mathcal{P}, \end{aligned} \quad (79)$$

$$d\mu = dx dp_x dy dp_y d\mu^0 d\pi_0 d\mathcal{P} d\bar{\mathcal{C}} d\bar{\mathcal{P}} d\bar{\mathcal{C}}, \quad (80)$$

where it is possible to establish variations and to compare between Fermionic conditions Ψ_0 and Ψ_1 . As the starting point we establish a infinitesimal difference that is

$$\chi_0 = i \int_{t_1}^{t_2} d\tau (\Psi_1 - \Psi_0) = i \int_{t_1}^{t_2} d\tau \bar{\mathcal{C}}(x - U^{\frac{1}{2}}(x)). \quad (81)$$

With this above difference is possible to establish a transformation that change variables resulting

$$\begin{aligned} x' &= x + \{x, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = x + \mathcal{C}\chi_0, \\ y' &= y + \{y, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = y + i\mathcal{C}\chi_0, \\ \mu'^0 &= \mu^0 + \{\mu^0, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = \mu^0 - i\mathcal{P}\chi_0, \\ p_{x'} &= p_x + \{p_x, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = p_x, \\ p_{y'} &= p_y + \{p_y, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = p_y, \\ \pi'_0 &= \pi_0, \\ -i\mathcal{P}' &= -i\mathcal{P} + \{-i\mathcal{P}, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = -i\mathcal{P}, \end{aligned}$$

$$\begin{aligned}
\mathcal{C}' &= \mathcal{C} + \{\mathcal{C}, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = \mathcal{C} \\
\bar{\mathcal{C}}' &= \bar{\mathcal{C}} + \{\bar{\mathcal{C}}, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\}\chi_0 = \bar{\mathcal{C}} + i\chi_0\pi_0, \\
\bar{\mathcal{P}}' &= \bar{\mathcal{P}} + \chi_0\{\bar{\mathcal{P}}, (\mathcal{C}\Phi_0 - i\mathcal{P}\pi_0)\} = \bar{\mathcal{P}} - \chi_0\Phi_0,
\end{aligned} \tag{82}$$

and that is applied to the BRST Hamiltonian

$$H'_{BRST} = \frac{1}{2}p_x^2 + \frac{\omega^2}{2}(x + iy)^2 = H_{BRST}, \tag{83}$$

that is invariant under the gauge transformations (82).

On another side, the kinetic term is invariant under these BRST transformations

$$\begin{aligned}
\dot{x}'p'_x + \dot{y}'p'_y - \mu'^0\dot{\pi}'_0 + \bar{\mathcal{C}}'\dot{\mathcal{P}}' + \bar{\mathcal{P}}'\dot{\mathcal{C}}' &= \\
\dot{x}p_x + \dot{y}p_y - \mu^0\dot{\pi}_0 + \bar{\mathcal{C}}\dot{\mathcal{P}} + \bar{\mathcal{P}}\dot{\mathcal{C}}.
\end{aligned} \tag{84}$$

So, in order to show that the path integral is invariant only is necessary to analyze a part of the effective Hamiltonian and the measure of the path integral.

Because, the kinetic part is invariant now is necessary to find as the effective Hamiltonian changes. The variation that results from these transformations in the effective Hamiltonian is

$$\delta H_{eff} = \{(\Psi_1 - \Psi_0), \Omega\} = \mathcal{C}\bar{\mathcal{C}}(1 - \frac{\partial U^{\frac{1}{2}}}{\partial x}) + i\pi_0(x - U^{\frac{1}{2}})] \tag{85}$$

where were used the gauge conditions (70).

The change in the effective action is

$$\begin{aligned}
e^{i\delta S_{eff}} &= \exp\left(-i \int d\tau \{\Omega, (\Psi_1 - \Psi_0)\}\right) \\
&= \exp\left(-i \int d\tau [\mathcal{C}\bar{\mathcal{C}}(1 - \frac{\partial U^{\frac{1}{2}}}{\partial x}) + i\pi_0(x - U^{\frac{1}{2}})]\right),
\end{aligned} \tag{86}$$

where it is shown an extra term that is originated from the proposed change.

On another side, it is important to consider the measure of the path integral. Here we are considered an infinitesimal change in the variables produced by these gauge transformations

$$\begin{aligned}
\frac{\delta x'}{\delta x} &= \delta(t - t') + i\mathcal{C}\bar{\mathcal{C}}(1 - \partial_x U^{\frac{1}{2}}), \\
\frac{\delta \bar{\mathcal{C}}'}{\delta \bar{\mathcal{C}}} &= \delta(t - t') - \pi_0(x - U^{\frac{1}{2}}),
\end{aligned} \tag{87}$$

and other variations are a Dirac delta that depends on time. With these variations the change for the measure of the path integral is

$$\mathcal{D}\mu' = \mathcal{D}\mu \exp\left(i \int d\tau [\mathcal{C}\bar{\mathcal{C}}(1 - \frac{\partial U^{\frac{1}{2}}}{\partial x}) + i\pi_0(x - U^{\frac{1}{2}})]\right). \tag{88}$$

And now it is possible to consider all above elements and to apply in the path integral, in a way infinitesimal, in order to show that the path integral is invariant under the change generated by this symmetry

$$Z_{\Psi_0} = \int \mathcal{D}\mu \exp(iS_{eff}) = \int \mathcal{D}\mu' \exp(iS'_{eff}) = Z_{\Psi_1}. \tag{89}$$

It is important to mention that the BRST charge generates an infinitesimal change in the phase space that if we apply to the elements of the path integral (76). We observe a contribution that helps to pass from a Fermionic condition to another. So, we have shown the Fradkin-Vilkovisky theorem for this complex case and we have shown that both systems with gauge conditions γ_0 and γ_1 are connected. In the next Section, we calculate the path integral for the complex harmonic oscillator in one dimension.

5. Path integral and Gauge Conditions

It is a central point to establish a quantum description of this complex model and it is important to include the gauge freedom. A way to achieve this description is through the Senjanovic method that allows to quantize systems with second class constraints [14]. In a particular form, we return to harmonic oscillator in two dimensions (6) with gauge conditions γ_{0*} , γ_{1*} (42) and the first class constraints (10) that are introduced using the next notation

$$\xi^a = (x, y, u, v), \quad \rho_a = (p_x, p_y, p_u, p_v), \quad (90)$$

$$\tau_b = (\Phi_0, \gamma_{0*}, \Phi_1, \gamma_{1*}) \quad (91)$$

where ξ^a includes the real and imaginary parts of z_1 , z_2 and ρ_a that is a complex quantity for momenta.

From these elements now it is possible to present the path integral that includes first class constraints and gauge conditions. The integration measure is

$$\mathcal{D}\Xi = \mathcal{D}\xi^a \mathcal{D}\rho_a \det | \{ \tau_b, \tau_c \} | \delta(\Phi_0) \delta(\gamma_{0*}) \delta(\Phi_1) \delta(\gamma_{1*}), \quad (92)$$

and the total path integral is

$$Z = \int \mathcal{D}\Xi \exp[i \int dt (\dot{\xi}^a \rho_a - H_{2C})]. \quad (93)$$

From the path integral and these constraints, it is possible to eliminate degrees of freedom (y, p_y, v, p_v) , that are imaginary parts through the delta functions $\delta(\tau_C)$, resulting

$$Z_R = \int \mathcal{D}x \mathcal{D}p_x \mathcal{D}u \mathcal{D}p_u \prod_j |p_{xj} p_{uj}|^2 \exp[i \int dt (-p_x^2 \dot{x} - x p_x \dot{p}_x - p_u^2 \dot{u} - u p_u \dot{p}_u - H_{1Cons})] \quad (94)$$

that is the real reduced path integral.

It is interesting to mention as we have achieved to reduce a complex formalism in a real theory by means of a particular way of projecting. This projection is not unique and exists an infinite number of possibilities according to the gauge conditions selected.

6. Canonical quantization

To analyze the role played by this gauge symmetry in the canonical quantization we parametrize the action for the model introduced in (62). In this case the action is

transformed to

$$S = \int_{\tau_0}^{\tau_1} d\tau \left(\frac{\dot{z}^2}{2\dot{t}} - \frac{\omega^2}{2} \dot{t} z^2 \right) \quad (95)$$

in terms of real and imaginary parts we get

$$S = \int_{\tau_0}^{\tau_1} d\tau \left(\frac{(\dot{x} + i\dot{y})^2}{2\dot{t}} - \frac{\omega^2}{2} \dot{t} (x + iy)^2 \right) \quad (96)$$

Notice that we are assuming that the time variable t is a real variable. For the momenta we obtain

$$p_x = \frac{(\dot{x} + i\dot{y})}{\dot{t}}, \quad p_y = \frac{i(\dot{x} - \dot{y})}{\dot{t}}, \quad p_t = -\frac{(\dot{x} + i\dot{y})^2}{2\dot{t}^2} - \frac{\omega^2}{2} (x + iy)^2 \quad (97)$$

From these expressions we see that there are now two constraints

$$\Phi = p_x + ip_y \approx 0, \quad (98)$$

$$\Psi = p_t + \frac{p_x^2}{2} + \frac{\omega^2}{2} (x + iy)^2 \approx 0. \quad (99)$$

By computing the canonical Hamiltonian we see that vanishes then the total Hamiltonian is given by

$$H_T = \mu^1 \Phi + \mu_2 \Psi, \quad (100)$$

From the evolution of the constraints we find that there are not more constraints and that both constraints are first class. To quantize this system we will fix partially the gauge, using a gauge condition for the constraint Φ . For the other first class constraint Ψ , we will use Dirac's quantization condition

$$\hat{\Psi}|\psi\rangle = 0, \quad (101)$$

For a given basis $|z, t\rangle$, we get

$$\left(\hat{p}_t + \frac{\hat{p}_x^2}{2} + \frac{\omega^2}{2} (\hat{x} + i\hat{y})^2 \right) \psi(x + iy, t) = 0. \quad (102)$$

For example, fixing the gauge

$$\gamma_1 = y \approx 0, \quad (103)$$

we obtain the usual harmonic oscillator, with Dirac's brackets

$$\{x, p_x\}^* = 1, \quad (104)$$

and normalized eigenfunctions

$$\psi(x, t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\omega}{\pi \hbar} \right)^{1/4} H_n \left(\sqrt{\frac{\omega}{\hbar}} x \right) \exp \left(-\frac{\omega x^2}{2\hbar} \right) \exp \left(\frac{-i}{\hbar} E_n t \right) \quad (105)$$

But now, fixing a different gauge, for example

$$\gamma_2 = y - i(x - U^{1/2}(x)) \approx 0, \quad (106)$$

In this case the constraint Ψ is reduced to

$$\Psi = \left(\hat{p}_t + \frac{\hat{p}_x^2}{2} + \frac{\omega^2}{2} U(\hat{x}) \right) \approx 0, \quad (107)$$

and the Dirac bracket for the reduced space is given by

$$\{x, p_x\}^* = \frac{2U^{1/2}}{\partial_x U}, \quad (108)$$

we can check that the above bracket is equal to (104) in the case of $U(x) = x^2$. By promoting this bracket to commutators in the quantum theory, the momentum operator must be realized as

$$p_x = -i\hbar \frac{2U^{1/2}}{\partial_x U} \partial_x. \quad (109)$$

In consequence our Schrödinger equation is given explicitly by

$$\left(-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2} \left(\frac{2U^{1/2}}{\partial_x U} \partial_x \right)^2 + \frac{\omega^2}{2} U(\hat{x}) \right) \psi(U^{1/2}, t) = 0. \quad (110)$$

One can easily check that the eigenfunctions (105) with the substitution $x \rightarrow U^{1/2}$ are solutions of the equation (110) and in this sense the problem with the gauge (103) (the harmonic oscillator) is gauge related to the problem (110). In this form, we have shown that for different gauge choices the quantum problems are gauge related, also in the quantum canonical sense.

7. Discussion and Conclusions

In this work, we introduce a new symmetry for complex holomorphic systems. This invariance is a local symmetry that it is trivial when is written in terms of complex variables. However, by written it in terms of real and imaginary parts allows us to connect in the complex space, real systems that apparently are not connected.

The general idea of the proposed formalism is to write any real theory in terms of complex variables, then to show that there is a gauge symmetry of the system. The next step is to analyze this symmetry in the canonical formalism and show that associated with the gauge symmetry we have first class constraints. This implies that we need to fix the gauge, in consequence we can select any "good" canonical gauge to project from the complex space to the real space. For each of the gauge choices, we have a different theory in real space. However, each of these theories is related by a gauge transformation in the complex space. The validity interval of each gauge condition is determined by the Dirac theory of constraints [1]. Finally, we show using canonical quantization, that the solutions of the respective Schrödinger equations, of two systems connected by a gauge transformation are also gauge related.

In this way, in the present work we have shown that following the idea developed in [8], we can consider different types of gauge conditions that lead us to different real systems. Furthermore, that using the gauge transformations we can find the solutions of the systems that are gauge connected, both at the classical level as at the quantum level. In some sense our procedure, is a generalization of the canonical transformations. Since for a given gauge condition we can get a system with a different symplectic structure. Also, in some sense, is related to the two-time physics of Bars [15], where from a space

with two times we can get by using a gauge choice different one time systems. The essential difference is that in our case we have more freedom, since we can choose any real system, write it in complex coordinates and then project it in different ways to real space. Our procedure can be generalized to field theory and also to high order time derivative theories [16].

8. Acknowledgments

The authors acknowledge partial support from CONACYT project 237503 and DGAPA-UNAM grant IN 103716.

References

- [1] Dirac P A M 2001 Lectures on Quantum Mechanics (vol 151) ed Dover (New York) chapter 1 pp 1-25
- [2] Henneaux M and Teitelboim C 1992 Quantization of Gauge Systems ed Princeton University Press (Princeton)
- [3] Gribov V N 1978 Quantization of non-Abelian gauge theories *Nucl. Phys. B* **139** 1
- [4] Ashtekar A 1991 Lectures on Non-perturbative Canonical Gravity ed World Scientific (Singapore)
- [5] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
- [6] Moiseyev N 2011 Non-Hermitian Quantum Mechanics ed Cambridge U P (Cambridge)
- [7] Gilmore R 1974 Lie Groups, Lie Algebras and some of their applications ed J. Wiley (New York)
- [8] Margalli C A and Vergara J D 2015 Hidden Gauge Symmetry in Holomorphic Models *Phys.Lett. A* **379** 2434
- [9] Moshinsky M and Quesne C 1971 Linear Canonical Transformations and Their Unitary Representations *J. Math. Phys.* **12** 1772
- [10] Morales-Tecotl H A, Urrutia L F and Vergara J D 1996 *Class. Quant. Grav.* **13** 2933 *Preprint* gr-qc/9607044
- [11] R. Remmert 1991 Theory of Complex Functions ed Springer-Verlag (New York)
- [12] Henneaux M 1985 Hamiltonian form of the path integral for theories with gauge freedom *Phys. Rep.* **126** 1
- [13] Fradkin E S and Vilkovisky G A 1977 *CERN Report TH-2332*
- [14] Senjanovic P 1976 *Ann. of Phys.* **100** 227
- [15] Bars I 2001 *Class. Quantum Grav.* **18** 3113
- [16] Margalli C A, Vergara J D and Romero J M 2017 work in progress.